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The Scenario for the Astrophysics with Scalar Field and the Cosmological Constant

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Abstract

In order to give the standard scenario of the astrophysics, we study the Einstein theory with minimally coupled scalar field and the cosmological term by considering the scalar field as a candidate of the dark matter. We obtained the exact solution in the cosmological scale and the approximate gravitational potential in the galactic or solar scale. We find that the scalar field plays the role of the dark matter in some sense in the cosmological scale but it does not play the role of the dark matter in the galactic or solar scale within our approximation.

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1 Introduction

Recently the existence of the cosmological constant becomes quite probable from the observation of the deep galaxy survey [1, 2]. While the dark matter is necessary in various observations such as the rotational curve of the spiral galaxy or the missing of the ordinary matter in the cosmological scale.

Then we start from the theory of general relativity with the dark matter and the cosmological constant in order to study the standard scenario of the astrophysics, that is, the physics of the cosmological, the galactic or solar scale. Though the neutrino is the promising candidate of the dark matter, there is no established direct observation of the dark matter as the ordinary matter. There is another attempt to explain the rotation curves in the theory of the Brans-Dicke theory [3, 4], where the Newtonian force is modified by the effect of the scalar field. In this paper, we consider the scalar field as a candidate of the dark matter [5] together with the cosmological constant and study their effects on time development of the scale factor of the universe in the cosmological scale and the gravitational potential in the galactic or solar scale.

The famous scalar-tensor theory is the Brans-Dicke theory, but we adopt the Einstein theory with minimally coupled scalar field instead of the Brans-Dicke theory. Our principle of the choice of the theory is the following. For the scalar-tensor gravity theory, we can transform one from the Jordan frame to the Einstein frame by the conformal transformation [6]. We prefer to adopt the Einstein frame because the post-Newtonian test of the general relativity such as the radar echo delay is quite stringent [7, 8]. Also we prefer the Einstein theory with minimally coupled scalar field because the scalar field as the dark matter means that the scalar field has no direct interaction of the gravitational field with the ordinary matter.

2 Classical Solution with Minimally Coupled Scalar and Cosmological Constant

2.1 Einstein Theory vs. Brans-Dicke Theory

The Brans-Dicke theory [3, 4] is the typical scalar-tensor theory of the gravity. The action of the Brans-Dicke theory with the cosmological term is given by

$$I_{\rm BD} = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}\left(\xi\phi^2 R - 2\Lambda\phi^n\right) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \mathcal{L}_{\rm ordinary\ matter}\right].$$

Uehara-Kim [9] found the general solution for $n = 2$ and matter dominant case, and Fujii [10] has found the special solution for general n.

Putting $g_{\mu\nu}(x) = \Omega^{-2}(x) g_{*\mu\nu}(x)$ with $\Omega(x) = \sqrt{\xi}\phi(x)$, we obtain the following action [6]

$$I_{\rm BD} = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}\left(R_* - 2\Lambda e^{(n-4)\zeta\phi_*}\right) - \frac{1}{2}g_*^{\mu\nu}\partial_\mu\phi_*\partial_\nu\phi_* + \xi^{-2}e^{-4\zeta\phi_*}\mathcal{L}_{*\text{ordinary matter}}\right] ,$$

where $\phi = \exp(\zeta\phi_*)$ with $\zeta^{-1} = \sqrt{1/\xi + 3/4\pi G}$ and $\mathcal{L}_{*\text{ordinary matter}}$ is obtained from $\mathcal{L}_{\text{ordinary matter}}$ by replacing the metric part in the form $g_{\mu\nu} \to g_{\mu\nu}\xi^{-1}\exp(\zeta\phi_*)$.

Our philosophy to fix the theory comes from the following two principles: i) the kinetic part of the gravity is of the standard Einstein form, because of the stringent constraint of the post-Newtonian test such as the delay of the radar echo experiment, ii) the scalar field has no direct coupling to the ordinary matter nor gives the effect on the geodesic equation of the particle. From these principles, we do not adopt the Brans-Dicke theory. In the following, we adopt the Einstein theory with standard cosmological term and the minimally coupled scalar field.

2.2 Einstein Theory with Minimally Coupled Scalar Field

By considering the minimally coupled scalar field as some kind of dark matter, we study the prototype of the time development of the scale factor of the universe in the cosmological scale and the gravitational potential in the galactic or solar scale.

We use Misner-Thorne-Wheeler notation [11] and consider Einstein action with the cosmological constant, the minimally coupled scalar field and the ordinary matter

$$I = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}(R - 2\Lambda) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \mathcal{L}_{\text{ordinary matter}}\right] , \tag{2.1}$$

where G is the gravitational constant, R is the scalar curvature and ϕ is the minimally coupled scalar field. The equations of motion in this system are given by

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} &= 8\pi G(T^\phi_{\mu\nu} + T_{\mu\nu}), && (2.2) \\ \partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) &= 0, && (2.3) \end{aligned}$$

where $T^\phi_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi$ and $T_{\mu\nu}$ is the energy-momentum tensor of the ordinary matter.

3 Cosmological Exact Solution

In order to study classical solutions in cosmology, we substitute the homogeneous, isotropic and flat metric

$$ds^2 = -dt^2 + a(t)^2 \left[dr^2 + r^2(d\theta^2 + \sin\theta^2 d\varphi^2) \right] \ , \tag{3.1}$$

and the perfect fluid expression of the ordinary matter $T_{\mu\nu} = (\rho+p)u_\mu u_\nu + p g_{\mu\nu}$ into equations of motion. We denote ρ and p as the density and the pressure of the perfect fluid respectively and we can take $u_\mu = (1,0,0,0)$ in the co-moving system. Then equations of motion to be solved become

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{\Lambda}{3} = \frac{8\pi G}{3}\left(\rho + \frac{\dot{\phi}^2}{2}\right), \tag{3.2}$$

$$\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\ddot{a}}{a} - \Lambda = -8\pi G\left(p + \frac{\dot{\phi}^2}{2}\right), \tag{3.3}$$

$$\frac{\ddot{\phi}}{\dot{\phi}} + 3\frac{\dot{a}}{a} = 0. \tag{3.4}$$

We consider the perfect fluid characterized by $p = \gamma\rho$, and we obtain the conservation law of the ordinary matter density by taking the linear combination of Eqs.(3.2), (3.3) and (3.4). From Eq.(3.4), we have another conservation law. Then we have the following two conservation laws

$$\rho = \rho_0 a^{-3(1+\gamma)}, \tag{3.5}$$

$$\dot{\phi} = \frac{k}{a^3}, \tag{3.6}$$

where ρ_0 and k are integration constants. The equation to be solved becomes Eq.(3.2) with the conditions Eqs.(3.5) and (3.6). Substituting Eqs.(3.5) and (3.6) into Eq.(3.2), we have the equation of the form

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{\Lambda}{3} = \frac{8\pi G}{3}\left(\frac{\rho_0}{a^{3(1+\gamma)}} + \frac{k^2}{2a^6}\right). \tag{3.7}$$

As the mathematical problem, we can solve exactly in the $\gamma = 1$ and $\gamma = 0$ case, but $\gamma = 1$ case is unphysical. Then we consider only the $\gamma = 0$ case, that is, the matter dominant case. In this case, we obtain

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{4\pi G}{3}\left(\dot{\phi} + \frac{\rho_0}{k}\right)^2 = \frac{\Lambda}{3} - \frac{4\pi G {\rho_0}^2}{3k^2} \tag{3.8}$$

by using Eqs.(3.2), (3.5) and (3.6).

3.1 $k^2\Lambda > 4\pi G{\rho_0}^2$ case

In this case the cosmological and/or the scalar term are dominant, and we parametrize

$$\frac{\dot{a}}{a} = \sqrt{\frac{\Lambda}{3} - \frac{4\pi G{\rho_0}^2}{3k^2}}\cosh\Theta, \tag{3.9}$$

$$\dot{\phi} = -\frac{\rho_0}{k} + \sqrt{\frac{\Lambda}{4\pi G} - \frac{{\rho_0}^2}{k^2}}\sinh\Theta. \tag{3.10}$$

Substituting this parametrization into Eq.(3.4), we have

$$\dot{\Theta} + \sqrt{\frac{3\Lambda}{1+A^2}}(\sinh\Theta - A) = 0, \tag{3.11}$$

where $A^{-1} = \sqrt{k^2\Lambda/4\pi G{\rho_0}^2 - 1}$. Then we obtain

$$\frac{1}{\sqrt{1+A^2}}\log\left|\frac{-A\tanh(\Theta/2) - 1 + \sqrt{1+A^2}}{-A\tanh(\Theta/2) - 1 - \sqrt{1+A^2}}\right| = -\sqrt{\frac{3\Lambda}{1+A^2}}(t-t_0) \tag{3.12}$$

by using the formula

$$\int\frac{d\Theta}{\sinh\Theta - A} = \frac{1}{\sqrt{1+A^2}}\log\left|\frac{-A\tanh(\Theta/2) - 1 + \sqrt{1+A^2}}{-A\tanh(\Theta/2) - 1 - \sqrt{1+A^2}}\right|. \tag{3.13}$$

And then we have the relation

$$\frac{-A\tanh(\Theta/2) - 1 + \sqrt{1+A^2}}{-A\tanh(\Theta/2) - 1 - \sqrt{1+A^2}} = \exp\left(-\sqrt{3\Lambda}\,(t-t_0)\right), \tag{3.14}$$

which gives the relation

$$\tanh(\Theta/2) = -\frac{1}{A} + \frac{\sqrt{1+A^2}}{A\tanh\left(\sqrt{3\Lambda}\,(t-t_0)\,/2\right)}. \tag{3.15}$$

Using this relation, we can write $\cosh\Theta$ and $\sinh\Theta$ in the form

$$\cosh\Theta = \frac{1+\tanh^2(\Theta/2)}{1-\tanh^2(\Theta/2)} = \frac{(1+A^2)\cosh(T-T_0) - \sqrt{1+A^2}\sinh(T-T_0)}{-A^2 - \cosh(T-T_0) + \sqrt{1+A^2}\sinh(T-T_0)}, \tag{3.16}$$

$$\sinh\Theta = \frac{2\tanh(\Theta/2)}{1-\tanh^2(\Theta/2)} = \frac{A\left(1 - \cosh(T-T_0) + \sqrt{1+A^2}\sinh(T-T_0)\right)}{-A^2 - \cosh(T-T_0) + \sqrt{1+A^2}\sinh(T-T_0)}, \tag{3.17}$$

where $T = \sqrt{3\Lambda}t$ and $T_0 = \sqrt{3\Lambda}t_0$. Introducing Θ_0 through the relation $\cosh\Theta_0 = \sqrt{1+A^2}/A$, $\sinh\Theta_0 = 1/A$, we can simplify the above expression in the form

$$\cosh\Theta = \frac{\sqrt{1+A^2}\cosh(T-T_0-\Theta_0)}{\sinh(T-T_0-\Theta_0)-A}, \tag{3.18}$$

$$\sinh\Theta = A\left(1+\frac{(A+1/A)}{\sinh(T-T_0-\Theta_0)-A}\right). \tag{3.19}$$

Using Eqs.(3.9) and (3.18), we have

$$\begin{aligned}
\log a &= \int\frac{da}{a} = \sqrt{\frac{\Lambda}{3}-\frac{4\pi G{\rho_0}^2}{3k^2}}\int dt\cosh\Theta \\
&= \frac{1}{3}\sqrt{1-\frac{4\pi G{\rho_0}^2}{k^2\Lambda}}\sqrt{1+A^2}\int dT\frac{\cosh\left(T-T_0-\Theta_0\right)}{\sinh\left(T-T_0-\Theta_0\right)-A} \\
&= \frac{1}{3}\log|\sinh(T-T_0-\Theta_0)-A|+\text{const.},
\end{aligned} \tag{3.20}$$

where we use the relation $4\pi G{\rho_0}^2/k^2\Lambda = A^2/(1+A^2)$. Therefore we have

$$a(t) = a_0\Big(\sinh(T-T_0-\Theta_0)-A\Big)^{1/3}, \tag{3.21}$$

where a_0 is the constant.

Similarly, from Eqs.(3.10) and (3.19), we have

$$\begin{aligned}
\phi &= \int dt\left(-\frac{\rho_0}{k}+\sqrt{\frac{\Lambda}{4\pi G}-\frac{{\rho_0}^2}{k^2}}\sinh\Theta\right) = -\frac{\rho_0}{k\sqrt{3\Lambda}}\int dT\left(1-\frac{\sinh\Theta}{A}\right) \\
&= \frac{\rho_0(1+A^2)}{kA\sqrt{3\Lambda}}\int\frac{dT}{\sinh(T-T_0-\Theta_0)-A} \\
&= \phi_0+\frac{1}{\sqrt{12\pi G}}\log\left|\frac{A\tanh\Big((T-T_0-\Theta_0)/2\Big)+1-\sqrt{1+A^2}}{A\tanh\Big((T-T_0-\Theta_0)/2\Big)+1+\sqrt{1+A^2}}\right| \\
&= \phi_1+\frac{1}{\sqrt{12\pi G}}\log\left|\frac{\exp(T-T_0-\Theta_0)-A-\sqrt{1+A^2}}{\exp(T-T_0-\Theta_0)-A+\sqrt{1+A^2}}\right|,
\end{aligned} \tag{3.22}$$

where ϕ_0 is the constant and ϕ_1 is given by $\phi_1 = \phi_0 + \dfrac{1}{\sqrt{12\pi G}}\log\left|\dfrac{1+A-\sqrt{1+A^2}}{1+A+\sqrt{1+A^2}}\right|$.

The integration constant a_0 is not the independent integration constant but it can be expressed by k and ρ_0. From Eqs.(3.6), (3.21) and (3.22), we have

$$\begin{aligned}\dot{\phi} &= \frac{\rho_0(1+A^2)}{kA\Big(\sinh(T-T_0-\Theta_0)-A\Big)} \\ &= \frac{\rho_0(1+A^2){a_0}^3}{kAa^3} = \frac{k}{a^3}.\end{aligned} \tag{3.23}$$

which gives the relation $a_0^3 = k^2A/\rho_0(1+A^2)$. This can be written in the form

$$k^2\Lambda - 4\pi G{\rho_0}^2 = \frac{\Lambda^2{a_0}^6}{4\pi G}. \tag{3.24}$$

Then we can obtain

$$a_0 = \Big(4\pi G(\frac{k^2}{\Lambda} - \frac{4\pi G\rho_0^2}{\Lambda^2})\Big)^{1/6}. \tag{3.25}$$

Therefore we have the exact solution in the form

$$a(t) = \Big(4\pi G(\frac{k^2}{\Lambda} - \frac{4\pi G\rho_0^2}{\Lambda^2})\Big)^{1/6}\Big(\sinh(T-T_1)-A\Big)^{1/3}, \tag{3.26}$$

$$\phi(t) = \phi_1 + \frac{1}{\sqrt{12\pi G}}\log\left|\frac{\exp(T-T_1)-A-\sqrt{1+A^2}}{\exp(T-T_1)-A+\sqrt{1+A^2}}\right|, \tag{3.27}$$

where

$$T = \sqrt{3\Lambda}t, \quad T_1 = T_0 - \Theta_0 = \sqrt{3\Lambda}t_1 = \text{const.},$$
$$A^{-1} = \sqrt{\frac{k^2\Lambda}{4\pi G\rho_0^2} - 1}, \quad \phi_1 = \text{const.}.$$

3.2 $4\pi G{\rho_0}^2 > k^2\Lambda$ case

In this case the ordinary matter is dominant, and we parametrize

$$\frac{\dot{a}}{a} = \sqrt{\frac{4\pi G{\rho_0}^2}{3k^2} - \frac{\Lambda}{3}}\sinh\Theta, \tag{3.28}$$

$$\dot{\phi} = -\frac{\rho_0}{k} + \sqrt{\frac{{\rho_0}^2}{k^2} - \frac{\Lambda}{4\pi G}}\cosh\Theta. \tag{3.29}$$

Substituting this parametrization into Eq.(3.4), we have

$$\dot{\Theta} + \sqrt{\frac{3\Lambda}{B^2 - 1}}(\cosh\Theta - B) = 0, \tag{3.30}$$

where $B^{-1} = \sqrt{1 - k^2\Lambda/4\pi G{\rho_0}^2}$. Then we obtain

$$\frac{1}{\sqrt{B^2 - 1}} \log \left| \frac{1 - B + \sqrt{B^2 - 1}\tanh(\Theta/2)}{1 - B - \sqrt{B^2 - 1}\tanh(\Theta/2)} \right| = -\sqrt{\frac{3\Lambda}{B^2 - 1}}(t - t_0), \tag{3.31}$$

by using the formula

$$\int \frac{d\Theta}{\cosh\Theta - B} = \frac{1}{\sqrt{B^2 - 1}} \log \left| \frac{1 - B + \sqrt{B^2 - 1}\tanh(\Theta/2)}{1 - B - \sqrt{B^2 - 1}\tanh(\Theta/2)} \right|.$$

Then we have the relation

$$\frac{1 - B + \sqrt{B^2 - 1}\tanh(\Theta/2)}{1 - B - \sqrt{B^2 - 1}\tanh(\Theta/2)} = -\exp\left(-\sqrt{3\Lambda}(t - t_0)\right), \tag{3.32}$$

where we take the branch of the logarithm in such a way as the scale factor of the universe behaves as the power law in time at the very early age of the universe. Then we have the relation

$$\tanh(\Theta/2) = \sqrt{\frac{B - 1}{B + 1}} \tanh\left(\sqrt{3\Lambda}(t - t_0)/2\right). \tag{3.33}$$

Using this relation, we can write $\sinh\Theta$ and $\cosh\Theta$ in the form

$$\sinh\Theta = \frac{2\tanh(\Theta/2)}{1 - \tanh^2(\Theta/2)} = \frac{\sqrt{B^2 - 1}\sinh(T - T_0)}{\cosh(T - T_0) - B}, \tag{3.34}$$

$$\cosh\Theta = \frac{1 + \tanh^2(\Theta/2)}{1 - \tanh^2(\Theta/2)} = \frac{B\cosh(T - T_0) - 1}{\cosh(T - T_0) - B}, \tag{3.35}$$

where $T = \sqrt{3\Lambda}t$ and $T_0 = \sqrt{3\Lambda}t_0$.

Using Eqs.(3.28) and (3.34), we have

$$\begin{aligned} \log a &= \int \frac{da}{a} = \sqrt{\frac{4\pi G{\rho_0}^2}{3k^2} - \frac{\Lambda}{3}} \int dt \sinh\Theta \\ &= \frac{1}{3}\sqrt{\frac{4\pi G{\rho_0}^2}{k^2\Lambda} - 1} \int dT \frac{\sqrt{B^2 - 1}\sinh(T - T_0)}{\cosh(T - T_0) - B} \\ &= \frac{1}{3}\log\left|\cosh(T - T_0) - B\right| + \text{const.}, \end{aligned} \tag{3.36}$$

where we use the relation $4\pi G{\rho_0}^2/k^2\Lambda = B^2/(B^2-1)$. Therefore we have

$$a(t) = a_0\Big(\cosh(T-T_0) - B\Big)^{1/3}, \tag{3.37}$$

where a_0 is the constant and $(T-T_0) = \sqrt{3\Lambda}(t-t_0)$.

Similarly, from Eqs.(3.29) and (3.35), we have

$$\begin{aligned}
\phi &= \int dt \left(\sqrt{1 - \frac{k^2\Lambda}{4\pi G{\rho_0}^2}}\cosh\Theta - 1\right) = \frac{\rho_0}{k\sqrt{3\Lambda}}\int dT\left(\frac{\cosh\Theta}{B} - 1\right) \\
&= \frac{\rho_0(B^2-1)}{kB\sqrt{3\Lambda}}\int\frac{dT}{\cosh(T-T_0) - B} \\
&= \phi_0 + \frac{\rho_0\sqrt{B^2-1}}{kB\sqrt{3\Lambda}}\log\left|\frac{1 - B + \sqrt{B^2-1}\tanh\Big((T-T_0)/2\Big)}{1 - B - \sqrt{B^2-1}\tanh\Big((T-T_0)/2\Big)}\right| \\
&= \phi_1 + \frac{1}{\sqrt{12\pi G}}\log\left|\frac{\exp(T-T_0) - B - \sqrt{B^2-1}}{\exp(T-T_0) - B + \sqrt{B^2-1}}\right|,
\end{aligned} \tag{3.38}$$

where ϕ_0 is the constant and ϕ_1 is given by $\phi_1 = \phi_0 + \dfrac{1}{\sqrt{12\pi G}}\log\left|\dfrac{1 + B - \sqrt{B^2-1}}{1 + B + \sqrt{B^2-1}}\right|$.

The integration constant a_0 is not the independent integration constant but it can be expressed by k and ρ_0. From Eqs.(3.6), (3.37) and (3.38), we have

$$\begin{aligned}
\dot{\phi} &= \frac{(B^2-1)\rho_0}{kB\Big(\cosh(T-T_0) - B\Big)} \\
&= \frac{(B^2-1)\rho_0{a_0}^3}{kBa^3} = \frac{k}{a^3},
\end{aligned} \tag{3.39}$$

which gives the relation $k^2 = (B^2-1)\rho_0{a_0}^3/B$. This can be written in the form

$$4\pi G{\rho_0}^2 - k^2\Lambda = \frac{\Lambda^2{a_0}^6}{4\pi G}. \tag{3.40}$$

Then we can obtain

$$a_0 = \Big(4\pi G\big(\frac{4\pi G{\rho_0}^2}{\Lambda^2} - \frac{k^2}{\Lambda}\big)\Big)^{1/6}. \tag{3.41}$$

Therefore we have the exact solution in the form

$$
\begin{aligned}
a(t) &= \Big(4\pi G(\frac{4\pi G{\rho_0}^2}{\Lambda^2}-\frac{k^2}{\Lambda})\Big)^{1/6}\Big(\cosh(T-T_0)-B\Big)^{1/3}, && (3.42)\\
\phi(t) &= \phi_0+\frac{1}{\sqrt{12\pi G}}\log\left|\frac{\exp(T-T_0)-B-\sqrt{B^2-1}}{\exp(T-T_0)-B+\sqrt{B^2-1}}\right|, && (3.43)
\end{aligned}
$$

where

$$
T=\sqrt{3\Lambda}t,\quad T_0=\sqrt{3\Lambda}t_0=\text{const.},
$$

$$
B^{-1}=\sqrt{1-\frac{k^2\Lambda}{4\pi G{\rho_0}^2}},\quad \phi_0=\text{const.}\ .
$$

3.3 $4\pi G{\rho_0}^2=k^2\Lambda$ case

For the completeness of the solution, we give the exact solution in this case. As the method to solve the equation is similar, we give only the result. The solution is given by

$$
\begin{aligned}
a(t) &= \Big(\frac{4\pi G\rho_0}{\Lambda}\Big)^{1/3}\Big(\exp(T-T_0)-1\Big)^{1/3}, && (3.44)\\
\phi(t) &= \phi_0+\frac{1}{\sqrt{12\pi G}}\log\Big|1-\exp\Big(-(T-T_0)\Big)\Big|, && (3.45)
\end{aligned}
$$

where

$$
T=\sqrt{3\Lambda}t,\quad T_0=\sqrt{3\Lambda}t_0=\text{const.},\quad \phi_0=\text{const.}\ .
$$

3.4 Special Limiting Case

i) $\rho_0=0$ case (no ordinary matter)

In case there is no ordinary matter $\rho_0\to 0$, which corresponds to $A\to\sqrt{4\pi G{\rho_0}^2/k^2\Lambda}$, we have the expression

$$
\begin{aligned}
a(t) &= \Big(\frac{4\pi Gk^2}{\Lambda}\Big)^{1/6}\Big(\sinh(T-T_1)\Big)^{1/3}, && (3.46)\\
\phi(t) &= \phi_1+\frac{1}{\sqrt{12\pi G}}\log\Big|\tanh\Big((T-T_1)/2\Big)\Big|. && (3.47)
\end{aligned}
$$

ii) $k=0$ case (no scalar matter): Lemaître universe [12]

In case there is no scalar matter $k \to 0$, which corresponds to $B \to 1$, we have the expression

$$
\begin{aligned}
a(t) &= \left(\frac{4\pi G\rho_0}{\Lambda}\right)^{1/3} \Big(\cosh(T-T_1)-1\Big)^{1/3}, && (3.48)\\
\phi(t) &= \phi_1. && (3.49)
\end{aligned}
$$

iii) $\Lambda=0$ case (no cosmological constant)

In case there is no cosmological term $\Lambda \to 0$, which corresponds to $B \to 1+k^2\Lambda/8\pi G{\rho_0}^2$, we have the expression

$$
\begin{aligned}
a(t) &= \lim_{\Lambda\to 0}\left(\frac{4\pi G\rho_0}{\Lambda}\right)^{1/3}\left(\frac{3\Lambda\,(t-t_0)^2}{2}-\frac{k^2\Lambda}{8\pi G{\rho_0}^2}\right)^{1/3}\\
&= (6\pi G\rho_0)^{1/3}\left((t-t_0)^2-\frac{k^2}{12\pi G{\rho_0}^2}\right)^{1/3} && (3.50)\\
\phi(t) &= \lim_{\Lambda\to 0}\left\{\phi_0+\frac{1}{\sqrt{12\pi G}}\log\left|\frac{\sqrt{3\Lambda}(t-t_0)-k\sqrt{\Lambda}/\sqrt{4\pi G\rho_0^2}}{\sqrt{3\Lambda}(t-t_0)+k\sqrt{\Lambda}/\sqrt{4\pi G\rho_0^2}}\right|\right\}\\
&= \phi_0+\frac{1}{\sqrt{12\pi G}}\log\left|\frac{(t-t_0)-k/\sqrt{12\pi G\rho_0^2}}{(t-t_0)+k/\sqrt{12\pi G\rho_0^2}}\right|. && (3.51)
\end{aligned}
$$

4 Effect on the Gravitational Potential

In this section, we calculate the effect of the cosmological constant and the scalar matter to the gravitational potential. For the special case of i) no scalar matter or ii) no cosmological term, the exact solutions are well-known.

4.1 Exact solution for special cases

i) No scalar matter case

In this case, we take the standard metric in the form

$$
ds^2 = -h(r)^2dt^2 + f(r)^2dr^2 + \Big[r^2(d\theta^2+\sin\theta^2 d\varphi^2)\Big] \ , \qquad (4.1)
$$

and the exact solution is given by [13]

$$h(r)^2 = 1 - r_0/r - \Lambda r^2/3, \tag{4.2}$$

$$f(r)^2 = \frac{1}{1 - r_0/r - \Lambda r^2/3}. \tag{4.3}$$

<u>ii) No cosmological term case</u>

In this case, we take the isotropic metric in the form

$$ds^2 = -h_1(r)^2 dt^2 + f_1(r)^2 \left[dr^2 + r^2(d\theta^2 + \sin\theta^2 d\varphi^2) \right] \ , \tag{4.4}$$

and the exact solution is given by [14]

$$\phi(r) = \phi_0 \log\left(\frac{r - r_0}{r + r_0}\right), \tag{4.5}$$

$$h_1(r)^2 = \left(\frac{r - r_0}{r + r_0}\right)^{2C}, \tag{4.6}$$

$$f_1(r)^2 = \left(1 - \frac{{r_0}^2}{r^2}\right)^2 \left(\frac{r + r_0}{r - r_0}\right)^{2C}, \tag{4.7}$$

where

$$\phi_0 = \sqrt{\frac{2(1 - C^2)}{8\pi G}}, \quad C = \text{const.} \ .$$

4.2 Cosmological term and scalar matter co-existing case

When the cosmological term and scalar matter co-exist, we cannot solve analytically, and we calculate the effect on the gravitational potential approximately. For this purpose, we take the standard metric Eq.(4.1) and the equations of motion Eqs.(2.2) and (2.3) are given by

$$\frac{4rf'}{f^2} + 2f - \frac{2}{f} - 2\Lambda r^2 f = \frac{8\pi r^2 G {\phi'}^2}{f}, \tag{4.8}$$

$$-\frac{4rh'}{f^2} - \frac{2h}{f^2} + 2h - 2\Lambda r^2 h = -\frac{8\pi r^2 G h {\phi'}^2}{f^2}, \tag{4.9}$$

$$\left(\frac{r^2 h \phi'}{f}\right)' = 0. \tag{4.10}$$

These can be rewritten into the form

$$
\begin{aligned}
\frac{f'}{f}-\frac{h'}{h}+\frac{f^2}{r}-\frac{1}{r} &= \Lambda r f^2, && (4.11)\\
\frac{f'}{f}+\frac{h'}{h} &= 4\pi G r \phi'^2, && (4.12)\\
\left(\frac{r^2 h\phi'}{f}\right)' &= 0. && (4.13)
\end{aligned}
$$

From Eq.(4.13), we have $\phi' = \alpha f/r^2 h$ with constant α. Then we have

$$
\begin{aligned}
(\log f/h)' = \frac{f'}{f}-\frac{h'}{h} &= -\frac{f^2}{r}+\frac{1}{r}+\Lambda r f^2, && (4.14)\\
(\log fh)' = \frac{f'}{f}+\frac{h'}{h} &= \frac{4\pi G\alpha^2 f^2}{r^3 h^2}. && (4.15)
\end{aligned}
$$

We introduce the new variables X, Y in the form $\exp(X) = fh$, $\exp(Y) = f/h$, then the above equation becomes in the form

$$
\begin{aligned}
Y' &= -\frac{\exp(X+Y)}{r}+\frac{1}{r}+\Lambda r\exp(X+Y), && (4.16)\\
X' &= \frac{4\pi G\alpha^2\exp(2Y)}{r^3}. && (4.17)
\end{aligned}
$$

For $\alpha = 0$ case (no scalar matter case), we have the solution

$$
X = 0, \quad \exp(Y) = \frac{1}{1-r_0/r-\Lambda r^2/3}, \tag{4.18}
$$

which is the exact solution to Eqs.(4.2) and (4.3). Then we calculate the gravitational potential by considering the region of r where $r_0/r, \Lambda r^2, 4\pi G\alpha^2/r^2 \ll 1$. In this approximation, we have

$$
\begin{aligned}
Y &\approx \frac{r_0}{r}+\frac{\Lambda r^2}{3}, && (4.19)\\
X &\approx 2\pi G\alpha^2\left(\frac{1}{{r_1}^2}-\frac{1}{r^2}\right) && (4.20)
\end{aligned}
$$

from Eqs.(4.17) and (4.18) where r_1 is constant.

In order to find the solution for $\alpha \neq 0$, we put $r_0 (= \text{const.}) \to r_0(r)$ (function of r). Then Eq.(4.16) becomes

$$
\begin{aligned}
Y' &= \left(\frac{r_0(r)}{r} + \frac{\Lambda r^2}{3} \right)' = -\frac{r_0}{r^2} + \frac{{r_0}'}{r} + \frac{2\Lambda r}{3} \\
&\approx -\frac{r_0}{r^2} + \frac{2\Lambda r}{3} - \frac{X}{r}
\end{aligned}
\tag{4.21}
$$

which gives ${r_0}' = -X$. Using Eq.(4.20), we have

$$
r_0(r) = r_2 - 2\pi G \alpha^2 \left(\frac{r}{{r_1}^2} + \frac{1}{r} \right), \tag{4.22}
$$

where r_2 is constant.

The gravitational potential Φ is given by

$$
\begin{aligned}
g_{00} &= -(1 + 2\Phi) = -\exp(X - Y) \\
&\approx -(1 - \frac{r_0(r)}{r} - \frac{\Lambda r^2}{3} + X) \\
&\approx -(1 + \frac{4\pi G \alpha^2}{r_1^2} - \frac{r_2}{r} - \frac{\Lambda r^2}{3}),
\end{aligned}
\tag{4.23}
$$

which gives $\Phi = 2\pi G \alpha^2 / r_1^2 - r_2/2r - \Lambda r^2/6$. Therefore the scalar matter does not contribute to the gravitational force $F_r = -\partial\Phi/\partial r = -r_2/2r^2 + \Lambda r/3$ within our approximation. The cosmological term contribute to the repulsive force within the approximation.

5 Summary and Discussion

We consider the scalar field as the candidate of the dark matter. Then, in order to give the standard scenario of the astrophysics, we study the Einstein theory with minimally coupled scalar field and the cosmological constant. We have studied various classical solutions with minimally coupled scalar and the cosmological term in the cosmological, the galactic or solar scale. We obtained the exact solution in the cosmology scale, where the scale factor expand in the power law in the first beginning and then expand exponentially. In the galactic or solar scale, we cannot find the exact solution, and examine the contribution from the scalar field to the gravitational potential and find that the scalar field does not contribute to the gravitational force within our approximation. In this way, in the cosmological scale, the scalar field play the role of the dark matter in some sense. While, in the galactic or solar scale, the scalar field does not pay the role of the dark matter.

For the ordinary matter, we first start from the classical Lagrangian and quantize the field and treat it as the classical smeared matter and make the perfect fluid approximation. While, in our approach, we treat the scalar field as the classical field in the same level as the classical gravitational field. If the metric is homogeneous, it may give the same effect whether we treat the scalar field as the classical field or the quantized and classically smeared matter. But if the metric is not homogeneous and is space-dependent, there is the quite big gap in the step of the quantization and the treatment of the classically smeared matter. In this sense, the scalar field may give the contribution to the gravitational force if we treat the scalar field as the quantized matter field.

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References

[1] S. Perlmutter *et al.*, Nature **391**, 51 (1998);
Astrophys. J. **517**, 565 (1999).

[2] A.G. Riess *et al.*, Astronom. J. **116**, 1009 (1998).

[3] C. Brans and R.H. Dicke, Phys. Rev. **124**, 925 (1961).

[4] Y. Fujii, Phys. Rev. **D9**, 874 (1974).

[5] F.S. Guzmán and T.M. Matos, gr-qc/9810028;
F.S. Guzmán, T. Matos and H. Villegas-Brena, astro-ph/9811143;
T.M. Matos, F.S. Guzmán and H.A. Ureña-López, astro-ph/9980152.

[6] P. Jordan, Nature, **164**, 637 (1949); Z. Phys. **157**, 112 (1959).

[7] S. Weinberg, *Gravitation and Cosmology* (John Wiley & Sons, Inc., New York, 1972).

[8] M. Kenmoku, Y. Okamoto and K. Shigemoto, Phys. Rev. **D48**, 578 (1993).

[9] K. Uehara and C.W. Kim, Phys. Rev. **D26**, 2575 (1982).

[10] Y. Fujii, Progr. Theor. Phys. **99**, 599 (1998).

[11] C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation* (W.H. Freeman and Company, San Francisco, 1973)

[12] G. Lemaître, Ann. Soc. Sci. Bruxelles, **A47**, 49 (1927).

[13] F. Kottler, Annalen der Physik **56**, 410 (1918).

[14] H.A. Buchdahl, Phys. Rev. **115**, 1325 (1959);
M. Wyman, Phys. Rev. **D24**, 839 (1981);
B.C. Xanthopoulos and T. Zannias, Phys. Rev. **D40**, 2564 (1989).